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ON THE SUMMABILITY OF THE DISCRETE HILBERT TRANSFORM

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Abstract: In this paper, we study the asymptotic behavior of the distribution function of the discrete Hilbert transform of sequences from the class l_1 and find a necessary condition and a sufficient condition for the summability of the discrete Hilbert transform of a sequence from the class l_1.

Keywords: Discrete Hilbert transform, Asymptotic behavior of the distribution function, Class of summable sequences.

Introduction

Denote by l_p, $p \geq 1$, the class of numeric sequences $b = \{b_n\}_{n\in Z}$ satisfying the condition

$$\|b\|_{l_p} = \Big(\sum_{n\in Z} |b_n|^p\Big)^{1/p} < \infty,$$

where Z is the set of integers.

Let $b = \{b_n\}_{n\in Z} \in l_1$. The sequence $H(b) = \{(Hb)_n\}_{n\in Z}$ is called the Hilbert transform of the sequence $b = \{b_n\}_{n\in Z}$, where

$$(Hb)_n = \sum_{m\neq n} \frac{b_m}{n-m}, \quad n \in Z.$$

M. Riesz proved (see [10] and [4, 7]) that, if $b \in l_p$, $p > 1$, then $H(b) \in l_p$ and the inequality

$$\|H(b)\|_{l_p} \leq C_p \|b\|_{l_p} \tag{0.1}$$

holds. Weighted analogues of (0.1) were investigated in [1–3, 5, 6, 8, 9, 11].

If $b \in l_1$, then the sequence $H(b)$ belongs to the class $\bigcap_{p>1} l_p$ but doesn't belong to the class l_1. In this case, R. Hunt, B. Muckenhoupt, and R. Wheeden proved (see [6]) that the distribution function

$$(Hb)(\lambda) \equiv \sum_{\{n\in Z:\ |(Hb)_n|>\lambda\}} 1$$

of the Hilbert transform of the sequence b satisfies the condition

$$\forall\lambda > 0 \quad |(Hb)(\lambda)| \le \frac{C_0}{\lambda} \|b\|_{l_1}, \tag{0.2}$$

where C_0 is an absolute constant.

In this paper, we study the asymptotic behavior of the distribution function $(Hb)(\lambda)$ of the Hilbert transform of a sequence $b \in l_1$ as $\lambda \to 0$ and find a necessary condition and a sufficient condition for the summability of the discrete Hilbert transform of a sequence from the class l_1.

1. Asymptotic behavior of the distribution function of the discrete Hilbert transform

Theorem 1. *Let $b \in l_1$. Then the following equation holds*:

$$\lim_{\lambda\to 0+} \lambda \cdot (Hb)(\lambda) = 2\Big|\sum_{n\in Z} b_n\Big|. \tag{1.1}$$

We first prove an auxiliary lemma.

Lemma 1. *Let $b \in l_1$ and $\sum_{n\in Z} b_n = 0$. Then the following equation holds*:

$$(Hb)(\lambda) = o(1/\lambda), \quad \lambda \to 0+. \tag{1.2}$$

P r o o f. Assume first that the sequence $b \in l_1$ is concentrated on some finite interval $[-m, m]$, i. e., $b_n = 0$ for $|n| > m$. In this case, from the equality

$$(Hb)_n = \sum_{|k|\le m} \frac{b_k}{n-k} - \frac{1}{n-1/2}\sum_{|k|\le m} b_k = \sum_{|k|\le m} \frac{k-1/2}{(n-k)(n-1/2)} b_k, \quad |n| > m$$

we get that

$$|(Hb)_n| \le \frac{4}{n^2}\sum_{|k|\le m}(k-1/2)\, b_k$$

for large values of n, whence the asymptotic equation (1.2) follows.

Let us now consider the general case. From the condition $\sum_{n\in Z} b_n = 0$, it follows that, for all $\varepsilon > 0$ there exist sequences $b' = \{b'_n\}_{n\in Z} \in l_1$ and $b'' = \{b''_n\}_{n\in Z} \in l_1$ satisfying the condition $b = b' + b''$, where the sequence $b' \in l_1$ is concentrated on some finite interval $[-m, m]$ and $\sum_{n\in Z} b'_n = 0$, and the sequence $b'' \in l_1$ satisfies the inequality $\sum_{n\in Z} |b''_n| < \varepsilon/(4C_0)$, with the constant C_0 from (0.2). Since the sequence $b' \in l_1$ is concentrated on $[-m, m]$ and $\sum_{n\in Z} b'_n = 0$, equation (1.2) is satisfied for the sequence $b' \in l_1$, and, therefore, there exists $\lambda(\varepsilon) > 0$ such that the inequality

$$\lambda\left(Hb'\right)(\lambda/2) < \varepsilon/2 \tag{1.3}$$

holds for $0 < \lambda < \lambda(\varepsilon)$, where $(Hb')(\lambda) = \sum_{\{n\in Z:\ |(Hb')_n| > \lambda\}} 1$. On the other hand, inequality (0.2) implies that

$$\lambda\left(Hb''\right)(\lambda/2) \le 2C_0 \sum_{n\in Z} \left|b''_n\right| < \varepsilon/2 \tag{1.4}$$

for all $\lambda > 0$, where $(Hb'')(\lambda) = \sum_{\{n\in Z:\ |(Hb'')_n| > \lambda\}} 1$. From inequalities (1.3) and (1.4) and the inclusion

$$\{n \in Z:\ |(Hb)_n| > \lambda\} \subset \{n \in Z:\ \left|(Hb')_n\right| > \lambda/2\} \bigcup \{n \in Z:\ \left|(Hb'')_n\right| > \lambda/2\}$$

we obtain that

$$\lambda \cdot (Hb)(\lambda) \le \lambda \left(Hb'\right)(\lambda/2) + \lambda \left(Hb''\right)(\lambda/2) < \varepsilon$$

for $0 < \lambda < \lambda(\varepsilon)$. This shows that equality (1.2) holds for all $b \in l_1$ satisfying the condition $\sum_{n\in Z} b_n = 0$. This completes the proof of Lemma 1. $\square$

P r o o f of Theorem 1. In the case $\sum_{n\in Z} b_n = 0$, the statement of the theorem follows from Lemma 1. Consider the case $\sum_{n\in Z} b_n = \alpha \ne 0$. We use the following notation: $b'_n = b_n$ for $n \ne 0$, $b'_0 = b_0 - \alpha$, $b''_n = 0$ for $n \ne 0$, and $b''_0 = \alpha$. Then $b = b' + b''$, where $b' = \{b'_n\}_{n\in Z} \in l_1$ and $b'' = \{b''_n\}_{n\in Z} \in l_1$. Since $\sum_{n\in Z} b'_n = 0$, we obtain from Lemma 1 that

$$(Hb')(\lambda) = o(1/\lambda), \quad \lambda \to 0+. \tag{1.5}$$

Since $(Hb''t)_n = \alpha/n$ for $n \ne 0$ and $(Hb'')_0 = 0$, we have

$$(Hb'')(\lambda) \sim \frac{2|\alpha|}{\lambda}, \quad \lambda \to 0+. \tag{1.6}$$

For all $0 < \varepsilon < 1$, by the inclusions

$$\left\{n \in Z : \left|\left(Hb''\right)_n\right| > (1+\varepsilon)\lambda\right\} \setminus \left\{n \in Z : \left|\left(Hb'\right)_n\right| > \varepsilon\lambda\right\} \subset$$
$$\subset \left\{n \in Z : \left|(Hb)_n\right| > \lambda\right\} \subset$$
$$\subset \left\{n \in Z : \left|\left(Hb'\right)_n\right| > \varepsilon\lambda\right\} \bigcup \left\{n \in Z : \left|\left(Hb''\right)_n\right| > (1-\varepsilon)\lambda\right\}$$

and relations (1.5) and (1.6), we have

$$\frac{2|\alpha|}{1+\varepsilon} \le \liminf_{\lambda\to 0+} \lambda \cdot (Hb)(\lambda) \le \limsup_{\lambda\to 0+} \lambda \cdot (Hb)(\lambda) \le \frac{2|\alpha|}{1-\varepsilon}.$$

This implies equation (1.1) and completes the proof of Theorem 1. $\square$

2. A necessary condition and a sufficient condition for the summability of the discrete Hilbert transform

Theorem 2. *Let $b \in l_1$. If $Hb \in l_1$, then it is necessary that the following equation holds:*

$$\sum_{n\in Z} b_n = 0. \tag{2.1}$$

P r o o f. We first we prove that, if $h = \{h_n\}_{n\in Z} \in l_1$, then the distribution function $h(\lambda) = \sum_{\{n\in Z:\, |h_n|>\lambda\}} 1$ of the sequence h satisfies the condition

$$h(\lambda) = o(1/\lambda), \quad \lambda \to 0+. \tag{2.2}$$

Note that the condition $h = \{h_n\}_{n\in Z} \in l_1$ implies that the set of $\{n \in Z : |h_n| > \lambda\}$ is finite for all $\lambda > 0$. Then, the inequality

$$\sum_{n\in Z} |h_n| = \sum_{\{n\in Z:\, |h_n|>1\}} |h_n| + \sum_{k=0}^{\infty} \left[\sum_{\left\{n\in Z:\, |h_n| \in \left(2^{-k-1};\, 2^{-k}\right]\right\}} |h_n| \right] \ge$$

$$\geq \sum_{\{n\in Z:\,|h_n|>1\}} 1 + \sum_{k=0}^{\infty}\left[\sum_{\{n\in Z:\,|h_n|\in\left(2^{-k-1};\,2^{-k}\right]\}} 2^{-k-1}\right] =$$

$$= h\left(1\right) + \sum_{k=0}^{\infty}\left[2^{-k-1}\cdot\left(h\left(2^{-k-1}\right) - h\left(2^{-k}\right)\right)\right] = \sum_{k=0}^{\infty}\left[2^{-k-1}\cdot h\left(2^{-k}\right)\right]$$

implies that

$$\lim_{k\to\infty} 2^{-k}\cdot h\left(2^{-k}\right) = 0.$$

Hence, taking into account that the function $h(\lambda)$ is decreasing, we obtain (2.2).

It follows from (2.1) that, if $Hb \in l_1$, then

$$\left(Hb\right)\left(\lambda\right) = o\left(1/\lambda\right), \quad \lambda \to 0+,$$

and, therefore, by Theorem 1, we obtain that the equation (2.2) holds. The proof of Theorem 2 is complete. □

Theorem 3. *If asequence $b \in l_1$ satisfies the conditions*
(i) $\sum\limits_{n\in Z} b_n = 0$;
(ii) $\sum\limits_{m\in Z} |b_m| \ln\left(e+|m|\right) < \infty$, *then $Hb \in l_1$ and the following inequality holds*:

$$\left\|Hb\right\|_{l_1} \leq 6\sum_{m\in Z}|b_m|\ln\left(e+|m|\right). \tag{2.3}$$

P r o o f. It follows from the definition of the discrete Hilbert transform that

$$\left|\left(Hb\right)_0\right| = \left|\sum_{m\neq 0}\frac{b_m}{m}\right| \leq \left\|b\right\|_{l_1}. \tag{2.4}$$

From condition (i) for $n \neq 0$, we obtain that

$$\left|\left(Hb\right)_n\right| = \left|\sum_{m\neq n}\frac{b_m}{n-m}\right| = \left|\sum_{m\neq n}\frac{b_m}{n-m} - \sum_{m\neq n}\frac{b_m}{n} - \frac{b_n}{n}\right| \leq \left|\frac{b_n}{n}\right| + \sum_{m\neq n}\frac{|m|\,|b_m|}{|n|\,|n-m|}. \tag{2.5}$$

It follows from inequalities (2.4) and (2.5) that

$$\left\|Hb\right\|_{l_1} = \sum_{n\in Z}\left|\left(Hb\right)_n\right| \leq 2\left\|b\right\|_{l_1} + \sum_{n\neq 0}\left[\sum_{m\neq n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\right] =$$

$$= 2\left\|b\right\|_{l_1} + \sum_{n>0}\left[\sum_{m>n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\right] + \sum_{n>0}\left[\sum_{m<n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\right] +$$

$$+ \sum_{n<0}\left[\sum_{m>n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\right] + \sum_{n<0}\left[\sum_{m<n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\right] =$$

$$= 2\left\|b\right\|_{l_1} + J_1 + J_2 + J_3 + J_4. \tag{2.6}$$

Let us estimate the summands J_k, $k = 1, 2, 3, 4$. From condition (ii) and f equalities of the form

$$\sum_{n<0}\left(\frac{1}{n-m} - \frac{1}{n}\right) = \left(\frac{1}{-1-m} + 1\right) + \left(\frac{1}{-2-m} + \frac{1}{2}\right) + ... + \left(\frac{1}{-m-m} + \frac{1}{m}\right) +$$

$$+\Big(\frac{1}{-m-1-m}+\frac{1}{m+1}\Big)+\Big(\frac{1}{-m-2-m}+\frac{1}{m+2}\Big)+...=1+\frac{1}{2}+...+\frac{1}{m},$$

for $m>0$, and

$$\sum_{n>0}\Big(\frac{1}{n}-\frac{1}{n-m}\Big)=\Big(1-\frac{1}{1+|m|}\Big)+\Big(\frac{1}{2}-\frac{1}{2+|m|}\Big)+...+\Big(\frac{1}{|m|}-\frac{1}{|m|+|m|}\Big)+$$

$$+\Big(\frac{1}{|m|+1}-\frac{1}{|m|+1+|m|}\Big)+\Big(\frac{1}{|m|+2}-\frac{1}{|m|+2+|m|}\Big)+...=1+\frac{1}{2}+...+\frac{1}{|m|},$$

for $m<0$, we obtain that

$$J_1=\sum_{n>0}\Big[\sum_{m>n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\Big]=\sum_{m>1}\Big[\sum_{0<n<m}\frac{m\,|b_m|}{n\,(m-n)}\Big]=$$

$$=\sum_{m>1}|b_m|\cdot\Big[\sum_{0<n<m}\Big(\frac{1}{m-n}+\frac{1}{n}\Big)\Big]=2\sum_{m>1}|b_m|\cdot\Big[1+\frac{1}{2}+...+\frac{1}{m-1}\Big]\le\sum_{m>1}|b_m|\cdot\ln m,$$

$$J_2=\sum_{n<0}\Big[\sum_{m>n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\Big]=\sum_{m>0}\Big[\sum_{n<0}\frac{m\,|b_m|}{n\,(n-m)}\Big]+\sum_{m<0}\Big[\sum_{n<m}\frac{m\,|b_m|}{n\,(m-n)}\Big]=$$

$$=\sum_{m>0}|b_m|\cdot\Big[\sum_{n<0}\Big(\frac{1}{n-m}-\frac{1}{n}\Big)\Big]+\sum_{m<0}|b_m|\cdot\Big[\sum_{n<m}\Big(\frac{1}{m-n}+\frac{1}{n}\Big)\Big]=$$

$$=\sum_{m>0}|b_m|\cdot\Big[1+\frac{1}{2}+...+\frac{1}{m}\Big]+\sum_{m<0}|b_m|\cdot\Big[1+\frac{1}{2}+...+\frac{1}{|m|}\Big]\le\sum_{m\in Z}|b_m|\cdot\ln\left(1+|m|\right),$$

$$J_3=\sum_{n>0}\Big[\sum_{m<n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\Big]=\sum_{m<0}\Big[\sum_{n>0}\frac{m\,|b_m|}{n\,(m-n)}\Big]+\sum_{m>0}\Big[\sum_{n>m}\frac{m\,|b_m|}{n\,(n-m)}\Big]=$$

$$=\sum_{m<0}|b_m|\cdot\Big[\sum_{n>0}\Big(\frac{1}{n}-\frac{1}{n-m}\Big)\Big]+\sum_{m>0}|b_m|\cdot\Big[\sum_{n>m}\Big(\frac{1}{n-m}-\frac{1}{n}\Big)\Big]=$$

$$=\sum_{m<0}|b_m|\cdot\Big[1+\frac{1}{2}+...+\frac{1}{|m|}\Big]+\sum_{m>0}|b_m|\cdot\Big[1+\frac{1}{2}+...+\frac{1}{m}\Big]\le\sum_{m\in Z}|b_m|\cdot\ln\left(1+|m|\right),$$

$$J_4=\sum_{n<0}\Big[\sum_{m<n}\frac{|m|\,|b_m|}{|n|\,|n-m|}\Big]=\sum_{m<-1}\Big[\sum_{m<n<0}\frac{m\,|b_m|}{n\,(n-m)}\Big]=$$

$$=\sum_{m<-1}|b_m|\cdot\Big[\sum_{m<n<0}\Big(\frac{1}{n-m}-\frac{1}{n}\Big)\Big]=$$

$$=2\sum_{m<-1}|b_m|\cdot\Big[1+\frac{1}{2}+...+\frac{1}{|m|-1}\Big]\le2\sum_{m<-1}|b_m|\cdot\ln|m|\,.$$

From this and (2.6), we obtain (2.3). The proof of Theorem 3 is complete. □

Theorem 4. *The following equation holds under the conditions of Theorem* 3:

$$\sum_{n\in Z}(Hb)_n=0. \tag{2.7}$$

P r o o f. By the conditions of Theorem 3,

$$(Hb)_0 = -\sum_{m\neq 0}\frac{b_m}{m}$$

and

$$(Hb)_n = \sum_{m\neq n}\frac{b_m}{n-m} = \sum_{m\neq n}\frac{b_m}{n-m} - \sum_{m\neq n}\frac{b_m}{n} - \frac{b_n}{n} = \sum_{m\neq n}\frac{mb_m}{n\,(n-m)} - \frac{b_n}{n}$$

for $n\neq 0$. Therefore, we have

$$\sum_{n\in Z}(Hb)_n = -\sum_{m\neq 0}\frac{b_m}{m} + \sum_{n\neq 0}\left[\sum_{m\neq n}\frac{mb_m}{n\,(n-m)} - \frac{b_n}{n}\right] = -2\sum_{m\neq 0}\frac{b_m}{m} + \sum_{n\neq 0}\left[\sum_{m\neq n}\frac{mb_m}{n\,(n-m)}\right] =$$

$$= -2\sum_{m\neq 0}\frac{b_m}{m} + \sum_{n>0}\left[\sum_{m>n}\frac{mb_m}{n\,(n-m)}\right] + \sum_{n>0}\left[\sum_{m<n}\frac{mb_m}{n\,(n-m)}\right] +$$

$$+\sum_{n<0}\left[\sum_{m>n}\frac{mb_m}{n\,(n-m)}\right] + \sum_{n<0}\left[\sum_{m<n}\frac{mb_m}{n\,(n-m)}\right] = -2\sum_{m\neq 0}\frac{b_m}{m} + j_1 + j_2 + j_3 + j_4. \tag{2.8}$$

It follows from condition (ii) that

$$j_1 = \sum_{n>0}\left[\sum_{m>n}\frac{mb_m}{n\,(n-m)}\right] = \sum_{m>1}\left[\sum_{0<n<m}\frac{mb_m}{n\,(n-m)}\right] =$$

$$= \sum_{m>1} b_m\cdot\left[\sum_{0<n<m}\left(\frac{1}{n-m} - \frac{1}{n}\right)\right] = -2\sum_{m>1} b_m\cdot\left[1+\frac{1}{2}+\ldots+\frac{1}{m-1}\right],$$

$$j_2 = \sum_{n<0}\left[\sum_{m>n}\frac{mb_m}{n\,(n-m)}\right] = \sum_{m>0}\left[\sum_{n<0}\frac{mb_m}{n\,(n-m)}\right] + \sum_{m<0}\left[\sum_{n<m}\frac{mb_m}{n\,(n-m)}\right] =$$

$$= \sum_{m>0} b_m\cdot\left[\sum_{n<0}\left(\frac{1}{n-m} - \frac{1}{n}\right)\right] + \sum_{m<0} b_m\cdot\left[\sum_{n<m}\left(\frac{1}{n-m} - \frac{1}{n}\right)\right] =$$

$$= \sum_{m>0} b_m\cdot\left[1+\frac{1}{2}+\ldots+\frac{1}{m}\right] - \sum_{m<0} b_m\cdot\left[1+\frac{1}{2}+\ldots+\frac{1}{|m|}\right],$$

$$j_3 = \sum_{n>0}\left[\sum_{m<n}\frac{mb_m}{n\,(n-m)}\right] = \sum_{m<0}\left[\sum_{n>0}\frac{mb_m}{n\,(n-m)}\right] + \sum_{m>0}\left[\sum_{n>m}\frac{mb_m}{n\,(n-m)}\right] =$$

$$= \sum_{m<0} b_m\cdot\left[\sum_{n>0}\left(\frac{1}{n-m} - \frac{1}{n}\right)\right] + \sum_{m>0} b_m\cdot\left[\sum_{n>m}\left(\frac{1}{n-m} - \frac{1}{n}\right)\right] =$$

$$= -\sum_{m<0} b_m\cdot\left[1+\frac{1}{2}+\ldots+\frac{1}{|m|}\right] + \sum_{m>0} b_m\cdot\left[1+\frac{1}{2}+\ldots+\frac{1}{m}\right],$$

$$j_4 = \sum_{n<0}\left[\sum_{m<n}\frac{mb_m}{n\,(n-m)}\right] = \sum_{m<-1}\left[\sum_{m<n<0}\frac{mb_m}{n\,(n-m)}\right] =$$

$$= \sum_{m<-1} b_m\cdot\left[\sum_{m<n<0}\left(\frac{1}{n-m} - \frac{1}{n}\right)\right] = 2\sum_{m<-1} b_m\cdot\left[1+\frac{1}{2}+\ldots+\frac{1}{|m|-1}\right].$$

From this and (2.8), we obtain (2.7). The proof of Theorem 4 is complete.

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